An Example of the Cauchy Problem with Infinitely Branching Solutions(Complex Analysis and Differential Equations)

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An Example of the Cauchy Problem with Infinitely Branching Solutions

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§1. Result

We consider in this note the following nonlinear first order Cauchy problem in $\mathbb{C}^2$

(1.1) $$\begin{cases} u_{x_1}{}^p + x_1{}^q + a(u_{x_2}{}^p + x_2{}^q) = 0 \\ u(0, x_2) = \phi(x_2) \end{cases}$$

with the following assumptions [A.1]−[A.3]:

[A.1] $p, q \in \mathbb{N}$ satisfying $p \geqq 2$, $q \geqq 2$, and $p + q \geqq 5$.

[A.2] ϕ is an entire function with the following property:

$\phi'(y)^p / y^q$ is not identically constant.

Notation 1.1. Let d be the greatest common divisor of p and q, and we put $p' := p/d$, $q' := q/d$. We denote the least common multiple by $m = dp'q'$. We put $\zeta_m := \exp(2\pi i/m)$ and denote the smallest field containing $\mathbb{Q} \cup \{\zeta_m\}$ by $\mathbb{Q}(\zeta_m)$.

Under this notation our third assumtion is as follows:

[A.3] a is a complex constant satisfying

$$\{\alpha \in \mathbb{C};\ \alpha^m = (-a)^{p'+q'}\} \cap \mathbb{Q}(\zeta_m) = \emptyset.$$

Note that [A.3] can be written as $(-a)^{1/p+1/q} \not\subset \mathbb{Q}(\zeta_m)$.

To state our result we give a remark on local holomorphic solutions of (1.1). We put

(1.2) $$F(x;\xi) := \xi_1{}^p + x_1{}^q + a(\xi_2{}^p + x_2{}^q)$$

$$g(x;\xi_2) := -x_1{}^q - a(\xi_2{}^p + x_2{}^q)$$

$$\psi(y) := \phi'(y)^p + y^q.$$

We fix a simply connected domain Ω satisfying

(1.3) $\quad \Omega \subset \mathbb{C}\setminus\psi^{-1}(0).$

Note that $\psi(y)/y^q \not\equiv$ const. by [A.2], in particular, $\psi(y)\not\equiv 0$, so there exists such a Ω satisfying (1.3). Since

$$g(0,y;\phi'(y)) = -a\,\psi(y) \neq 0 \text{ on } \Omega,$$

we can find a simply connected domain W in $\mathbb{C}^3$ containing the set $\{(0,y;\phi'(y));\ y\in\Omega\}$ so that $g(W)\cap\{0\}=\emptyset$. Then the equation $f^p=g$ has p holomorphic roots $f_k\in\mathcal{O}(W)$, $1\leqq k\leqq p$, so we get the following decomposition of F:

(1.4) $\quad F(x;\xi)=\prod_{k=1}^{p}(\xi_1 - f_k(x;\xi_2)) \quad (\xi_1\in\mathbb{C},\ (x;\xi_2)\in W).$

By virtue of (1.4) the Cauchy problem (1.1) can be reduced on a neighborhood $\Omega^{\sim}$ of $\{0\}\times\Omega$ in $\mathbb{C}^2$ to

$$(1.5)_k \quad \begin{cases} u_{x_1} = f_k(x;u_{x_2}) & x\in\Omega^{\sim} \qquad (1\leqq k\leqq p) \\ u(0,x_2)=\phi(x_2). \end{cases}$$

Notation 1.2. Since $(1.5)_k$ has a uniquely determined holomorphic solution on a neighborhood of $\{0\}\times\Omega$ in $\Omega^{\sim}$, we denote it by $u_k(x;\Omega)$. We also denote the maximal analytic continuation of $u_k(x;\Omega)$ to $\mathbb{C}^2$ by $u_k{}^*(x;\Omega)$.

Our main result is the following theorem:

Theorem 1.3. *Under* [A.1]−[A.3], *for any* Ω *and each* k, *the function* $u_k{}^*(x;\Omega)$ *is an infinitely many-valued function.*

Now we explain our motivation to consider (1.1). In [2] we studied first order nonlinear Cauchy problem in a neighborhood M of x^0 in $\mathbb{C}^n$:

$$(1.6) \quad \begin{cases} G(x;u_x;u)=0 & x=(x_1,\cdots,x_n)\in M \\ u(x_1{}^0,x')=\phi(x') & \text{on } M\cap\{x_1=x_1{}^0\}. \end{cases}$$

where $G(x;\xi;z)$ is holomorphic in the first order jet bundle $J^1(M)$ and where $\phi(x')$ is holomorphic on $M\cap\{x_1=x_1{}^0\}$. We fix a a point

$$e^0=(x^0;\xi_1{}^0,\phi_{x'}(x^0);\phi(x^0))\in J_{x^0}(M)$$

with $G(e^0)=0$, and assume the following [A.4]−[A.6]:

$$[A.4] \qquad \sum_j \frac{\partial G}{\partial \xi_j}(e^0)\frac{\partial}{\partial x_j} \neq 0 \quad \text{in } T_{x^0}(M).$$

[A.5] The function $\xi_1 \rightarrow G(x^0;\xi_1,\phi_{x'}(x^0);\phi(x^0))$ vanishes at ξ_1^0 with a finite vanishing order $p \geqq 1$.

[A.6] There exists a holomorphic extension $\Phi(x)$ of $\phi(x')$ with $\Phi_x(x^0)=(\xi_1^0,\phi_{x'}(x^0))$ to a neighborhood of x^0 in M so that Φ has several "good" properties.

For the precise meaning of [A.6] see § 2 in [2].

The main result of [2] was the following theorem.

Theorem 1.4 ([2], Theorem 4.2). *Under* [A.4]−[A.6], *the Cauchy problem (1.6) has finitely many-valued analytic solutions around* x^0. *Further, the ramification degrees of such solutions around* x^0 *can be calculable by means of the Newton polygon* $N(g^\Phi)$ *of the function*

$$g^\Phi(x';\tau)=G(x_1^0,x';\tau dx_1+\Phi_x(x_1^0,x');\phi(x')).$$

It is our motivation to consider (1.1) as an example of the Cauchy problem (1.6) which do *not* satisfy the condition [A.4]. Indeed, if we choose $\phi(x_2)$ in (1.1) as $\phi'(0)=0$ then [A.4] at the point $e^0=(0;0;\phi(0)) \in J^1(\mathbb{C}^2)$ is not satisfied. We also note that Theorem 1.3 concerns a *global* ramification degrees of the solutions of (1.1).

§2. Characteristic Strips

Our proof of Theorem 1.3 is based on the following three tools:

(I) Classical theory of characteristic strips for first order nonlinear Cauchy problems.

(II) Representation of the Hamilton flows associated with (1.1) by means of a special function $s_{pq}(\tau)$.

(III) Automorphic property of s_{pq} or its uniformization σ_{pq}.

In this section we give a quick review of (I). For the tools (II) and (III) see §§3−5.

Let us consider the following Cauchy problem in $\mathbb{C}^n$:

$$(2.1) \qquad \begin{cases} G(x;u_x;u)=0 & x \in \mathbb{C}^n \\ u(0,x')=\phi(x') & x' \in S=\{x_1=0\} \end{cases}$$

and assume that $G(x;\xi;z)$ and $\phi(x')$ are holomorphic. We also assume that there exists a domain Ω in S so that the equation

$$(2.2)\qquad G(0,y;\xi_1,\phi_{x'}(y);\phi(y))=0$$

has a holomorphic root $\xi_1=f(y)$ on Ω. We put

$$\rho(y)=(0,y;f(y),\phi_{x'}(y);\phi(y))$$

and denote the characteristic strip associated with G issuing from $\{\rho(y);y\in\Omega\}$ by $\Phi(t,y)=(X;\Xi;Z)(t,y)$:

$$(2.3)\qquad \partial_t X_j=[(\partial/\partial\xi_j)G](\Phi)\qquad (1\leqq j\leqq n)$$

$$\partial_t\Xi_j=-[(\partial/\partial x_j)G](\Phi)-\Xi_j[(\partial/\partial z)G](\Phi)\ (1\leqq j\leqq n)$$

$$\partial_t Z=\sum_{j=1}^{n}\Xi_j[(\partial/\partial\xi_j)G](\Phi)$$

$$(2.4)\qquad \Phi(0,y)=\rho(y).$$

Proposition 2.1. *If there exists a neighborhood* V *of* $(0,y_0)$ *in* $\mathbb{C}^n$ *so that the restriction* $X|_V$ *is biholomorphic, then the function* $u(x)=Z((X|_V)^{-1}(x))$, $x\in X(V)$, *is a holomorphic solution of* (2.1) *satisfying*

$$u_{x_1}(0,y)=f(y).$$

Further, the derivatives of u *are given by*

$$(2.5)\qquad u_{x_j}(x)=\Xi_j((X|_V)^{-1}(x)),\ 1\leqq j\leqq n$$

Proposition 2.1 follows from $G(\Phi(t,y))\equiv 0$ and from $\Phi^*\alpha=0$ where $\alpha=dz-\sum_{j=1}^{n}\xi_j dx_j$ is the fundamental 1-form on $J^1(\mathbb{C}^n)$.

§3. The Function s_{pq}

In this section we define and study a special function $s_{pq}(\tau)$ which represents the Hamilton flow associated with (1.1).

We first define a function $\tau_{pq}(s)$, and next define s_{pq} as the inverse function of it.

Definition 3.1. For $p,q\in\mathbb{N}$ satisfying [A.1] we define an open sector S_q and a function $\tau_{pq}(s)$ by

$$(3.1)\qquad S_q:=\{s\in\mathbb{C};\ 0<\arg(s)<\pi/q\},$$

$$(3.2)\qquad \tau_{pq}(s)=\int_{\Gamma(s)}(1-z^q)^{-(p-1)/p}dz,$$

where $\Gamma(s):[0,1] \to S_q \cup \{0\}$ be a path joinning 0 to $s \in S_q$. Choosing the branch of $(1-z^q)^{-1/p}$ at $z=0$ as $1^{-1/p}=1$, (3.2) determines a function $\tau_{pq} \in \mathcal{O}(S_q) \cap C(S_q^{\wedge})$ independently of the choice of a path $\Gamma(s)$, where $S_q^{\wedge}$ denotes the closure of S_q in the extended complex plane $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$.

Notation 3.2. We put $\omega_{pq} := \tau_{pq}(1) \in (0,\infty)$, and define an open triangle T_{pq} by

$$T_{pq} := \left\{ \tau \in \mathbb{C} \ ; \ \begin{array}{l} 0 < \arg(\tau) < \pi/q, \text{ and} \\ \pi(p-1)/p < \arg(\tau - \omega_{pq}) < \pi \end{array} \right\}. \tag{3.3}$$

We denote the closure of T_{pq} in $\mathbb{C}$ by T_{pq}^{-}, and the vertex of T_{pq} distinct from 0 and ω_{pq} by λ_{pq}.

Proposition 3.3. *The function* τ_{pq} *maps* S_q *conformally onto* T_{pq} *[resp. maps* $S_q^{\wedge}$ *homeomorphically onto* T_{pq}^{-}*], with*

$$(\tau_{pq}(0), \tau_{pq}(1), \tau_{pq}(\infty)) = (0, \omega_{pq}, \lambda_{pq}). \tag{3.4}$$

Proof. We use the formula of Schwarz-Christoffel, which asserts that the conformal map Ψ from the upper half plane H onto T_{pq} with the property $(\Psi(0), \Psi(1), \Psi(\infty)) = (0, \omega_{pq}, \lambda_{pq})$ can be written as the following form:

$$\Psi(z) = A \int_{z_0}^{z} \zeta^{-(q-1)/q} (\zeta - 1)^{-(p-1)/p} d\zeta + B \tag{3.5}$$

where $z_0 \in H$ and where A and B are constants. Substituting $z = \psi(s) = s^q$ into (3.5), we have the composition $\Psi_1 = \Psi \circ \psi$ given by

$$\Psi_1(s) = q A \int_{s_0}^{s} (\sigma^q - 1)^{-(p-1)/p} d\sigma + B \tag{3.6}$$

which maps S_q conformally onto T_{pq} with

$$(\Psi_1(0), \Psi_1(1), \Psi_1(\infty)) = (0, \omega_{pq}, \lambda_{pq}). \tag{3.7}$$

Evaluating the constants A and B by (3.7), we can deduce $\Psi_1 = \tau_{pq}$ so τ_{pq} has the desired property. □

Definition 3.4. We define $s_{pq}: T_{pq} \to S_q$ by $s_{pq} := \tau_{pq}^{-1}$. By Proposition 3.3 and (3.2), s_{pq} maps T_{pq} conformally onto S_q, with

$$\begin{cases} s_{pq}'(\tau) = [1 - s_{pq}(\tau)^q]^{(p-1)/p} \\ s_{pq}(0) = 0 \end{cases} \tag{3.8}$$

The following proposition is the key result in this section.

Proposition 3.5. *For* $p, q \in \mathbb{N}$ *satisfying* [A.1], *let* m *be the least common multiple of them (Notation 1.1). Then it follows that* $p^{-1}+q^{-1}+m^{-1} \leqq 1$. *Moreover, the following conditions* (a)–(d) *are equivalent:*
(a) s_{pq} *is an elliptic function on* $\mathbb{C}$.
(b) s_{pq} *is single-valued around the vertex* λ_{pq} *of* T_{pq}.
(c) *The equality* $p^{-1}+q^{-1}+m^{-1}=1$ *holds.*
(d) $(p,q) \in \{(2,3),(3,2),(2,4),(3,3),(4,2)\}$.

Proof. By [A.1], we have $1-p^{-1}-q^{-1}>0$, so $1-p^{-1}-q^{-1} \in m^{-1}\mathbb{N}$. Thus we get $p^{-1}+q^{-1}+m^{-1} \leqq 1$. Since (a) $\Rightarrow$ (b) is trivial, we only have to show (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (a).

To show (b) $\Rightarrow$ (c) we apply the Schwarz's reflection principle to s_{pq} around $\tau=\lambda_{pq}$. Let $T_{pq}{}^*$ be the reflection of T_{pq} with the segment $[\omega_{pq}, \lambda_{pq}]$, and let $S_q{}^*$ be the reflection of S_q with the half line $(1, \infty)$. Since $s_{pq}(T_{pq}{}^*)=S_q{}^*$ by the principle, if τ rotates $2(1-p^{-1}-q^{-1})\pi$ around λ_{pq} then the value $s_{pq}(\tau)$ rotates $2q^{-1}\pi$ around ∞. Thus the single-valuedness of s_{pq} yields that there exists a $n' \in \mathbb{N}$ so that

$$2(1-p^{-1}-q^{-1})\pi n'=2\pi.$$

So, if τ rotates 2π around λ_{pq} then $s_{pq}(\tau)$ rotates $2q^{-1}\pi n'$ around ∞, which deduces $q^{-1}n' \in \mathbb{N}$. Thus, there exists a $n \in \mathbb{N}$ satisfying $(1-p^{-1}-q^{-1})qn=1$. Then we deduce $qn/p \in \mathbb{N}$, which implies $n \in p'\mathbb{Z}$. Thus we get $qn \in m\mathbb{Z}$, so we conclude that $(1-p^{-1}-q^{-1})m$ is a divisor of 1, which shows (c).

To show (c) $\Rightarrow$ (d) we remark that (c) is equivalent to

(3.9) $$dp'q'=m=m(p^{-1}+q^{-1}+m^{-1})=q'+p'+1.$$

If $p'q'=1$ then $p'=q'=1$, so (3.9) implies $d=3$, thus we get $(p,q)=(3,3)$. If $p'q'=2$ then $(p',q')=(1,2)$ or $(2,1)$, so (3.9) implies $d=2$, thus we get $(p,q)=(2,4)$ or $(4,2)$. In the case $p'q' \geqq 3$, we use the inequality

(3.10) $$dp'q'=q'+p'+1 \leqq 2+p'q'$$

which is a consequence of $(p'-1)(q'-1) \geqq 0$. By (3.10) we have $0 \leqq d-1 \leqq 2/(p'q') \leqq 2/3$, so we get $d=1$. Then (3.9) means $pq=q+p+1$, which is equivalent to $(p-1)(q-1)=2$. Thus we deduce $(p,q)=(2,3)$ or $(3,2)$.

To show (d) $\Rightarrow$ (a), we need the following notation.

Notation 3.6. Let Q be the closure of the union of T_{pq} and its reflection with the real axis. Let r_k [resp. ρ] be the $2\pi/p$ [$2\pi/q$] rotation in $\mathbb{C}$ with center $\zeta_q{}^k\omega_{pq}$ [the origin]. We put

(3.11) $$F_{pq} := \text{the interior of } \Big[\bigcup_{j=0}^{p-1}\bigcup_{k=0}^{q-1} r_k{}^j(\rho^k(Q))\Big],$$

which is an open $2(p-1)q$-gon in $\mathbb{C}$. For each $n \in \mathbb{Z}$ satisfying $0 \leqq n \leqq (p-1)q-1$, dividing n by $(p-1)$, we write n as

(3.12) $$n = (p-1)k + j \quad (0 \leqq k, 0 \leqq j \leqq p-2),$$

and we define *sides* s_n and s_n' of F_{pq}, and *side-pairing maps*

$$\{g_s \in \mathrm{Aut}(\mathbb{C});\ s \in \{s_n, s_n';\ 0 \leqq n \leqq (p-1)q-1\}\}$$

as follows:

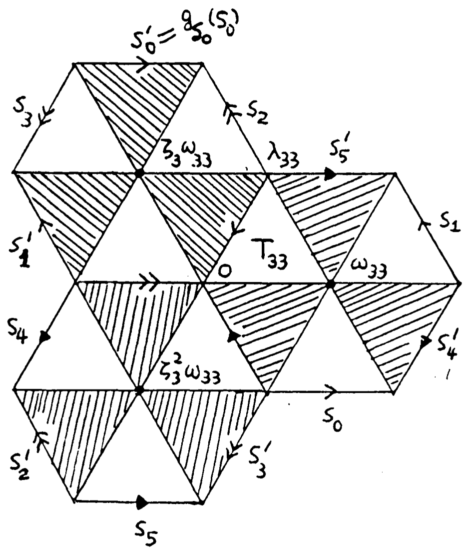
(3.13) $$s_n = r_k{}^{j+1}(\rho^k([0, \lambda_{pq}])),\quad s_n' = r_{k+1}{}^{j+1}(\rho^k([0, \lambda_{pq}]))$$

(3.14) $$g_s = r_{k+1}{}^{j+1} \circ r_k{}^{-j-1} \text{ if } s = s_n, \text{ and}$$

$$g_s = r_k{}^{j+1} \circ r_{k+1}{}^{-j-1} \text{ if } s = s_n'.$$

We illustrate the polygon F_{pq} in the case $(p, q) = (3, 3)$:

Figure 3.7. F_{33} with the side-pairing maps.



Let G_{pq} be the group generated by the side-pairing maps (3.14). Since any $g \in G_{pq}$ is a translation in $\mathbb{C}$, there exists a constant $c(g) \in \mathbb{C}$ so that $g(z) = z + c(g)$ for $z \in \mathbb{C}$. For all (p, q)

satisfying the condition (d), it can be verified that G_{pq} is generated by g_0 and g_1 which are independent over $\mathbb{R}$. Further, F_{pq} is a *fundamental domain* for the group G_{pq}, that is, the following conditions hold:

(3.15) $$\mathbb{C} = \bigcup_{g \in G_{pq}} g(F_{pq}^{-})$$

(3.16) $$g \in G_{pq} \setminus \{I\} \text{ implies } g(F_{pq}) \cap F_{pq} = \emptyset .$$

Since s_{pq} is invariant under the action of G_{pq} and is single-valued on $\mathbb{C}$, s_{pq} is an elliptic function. □

§4. Representation of Hamilton Flows

In this section we give the representation of the Hamilton flows associated with (1.1) by means of $s_{pq}(\tau)$. Let us recall the simply connected domain Ω in $\mathbb{C} \setminus \psi^{-1}(0)$ and the decomposition (1.4) of $F(x;\xi)$. We put

(4.1) $$\rho_k(y) = (0, y; f_k(0, y; \phi'(y)), \phi'(y))$$

which lies in $T^*(\mathbb{C}^2) \cap F^{-1}(0)$.

Lemma 4.1. *Let* Ω_1 *be a simply connected subdomain of* Ω *so that* $\{y^q/\psi(y); y \in \Omega_1\} \subset \mathbb{C} \setminus [1, \infty)$. *Then there exist* $A_k, B, E \in \mathcal{O}(\Omega_1)$ *such that*

$$A_k(y)^{q'} = f_k(0, y; \phi'(y))$$

$$B(y)^{p'} s_{pq}(E(y)) = y$$

$$B(y)^{q'} [1 - s_{pq}(E(y))^q]^{1/p} = \phi'(y).$$

Moreover, the function $\alpha_k(y) = -a(B(y)/A_k(y))^{m-p'-q'}$ *is constant on* Ω_1 *so that* $\alpha_k^m = (-a)^{p'+q'}$.

Proof. Since $f_k(0, y; \phi'(y))^p = -a\psi(y) \neq 0$ on Ω, the existence of A_k is obvious. To show the existence of B, we put

(4.2) $$V_q = \mathbb{C} \setminus \bigcup_{k=0}^{q-1} \zeta_q^k [1, \infty).$$

Let $B(y)$ be a holomorphic root of $B^m = \psi$ on Ω_1. Then, $y \in \Omega_1$ implies $(y/B^{p'})^q = y^q/\psi(y) \in \mathbb{C} \setminus [1, \infty)$, so $y/B^{p'} \in V_q$. Since $\tau_{pq}|V_q$ is single-valued, we can define $E_B \in \mathcal{O}(\Omega_1)$ by

$$E_B(y) = (\tau_{pq}|V_q)(y/B(y)^{p'}).$$

By $s_{pq} \circ \tau_{pq} = \mathrm{id}$, we get $B(y)^{p'} s_{pq}(E(y)) = y$ on Ω_1. Let us put $B^*(y) := \zeta_m B(y)$. Then

$$s_{pq}(E_{B*}(y)) = y/(\zeta_m B(y))^{p'} = s_{pq}(E_B(y))/\zeta_q$$

so we get $s_{pq}(E_{B*})^q = s_{pq}(E_B)^q = y^q/\psi(y) \in \mathbb{C} \setminus [1, \infty)$. Since $(1-z)^{-1/p}$ is single-valued on $\mathbb{C} \setminus [1, \infty)$, we deduce that

$$[1 - s_{pq}(E_{B*})^q]^{1/p} = [1 - s_{pq}(E_B)^q]^{1/p}.$$

Since $B^m = \psi$ implies

$$(B^{q'} [1 - s_{pq}(E_B)^q]^{1/p})^p = B^m(1 - s_{pq}(E_B)^q)$$

$$= \psi [1 - (y^q/\psi)] = \phi'(y)^p,$$

we can find some $j \in \{0, \cdots, m-1\}$ so that $B^{*(j)} = \zeta_m{}^j B$ satisfies

$$(B^{*(j)})^{q'} [1 - s_{pq}(E_{B*(j)})^q]^{1/p} = \phi'(y).$$

The last assertion on α_k easily follows from $(A_k/B)^m = -a$. □

The representation of the Hamilton flows is as follows:

Proposition 4.2. *Let* $\Phi_k = (X; \Xi)(t, y)$ *be the Hamilton flows associated with* $F(x;\xi) := \xi_1{}^p + x_1{}^q + a(\xi_2{}^p + x_2{}^q)$, *issuing from* $\rho_k(y)$ *at* $t = 0$. *Let* Ω_1 *and* $A_k, B, E \in \mathcal{O}(\Omega_1)$ *and* $\alpha_k \in \mathbb{C}$ *be chosen as in Lemma 4.1. Then* Φ_k *can be represented as*

$$X_1 = A_k(y)^{p'} s_{pq}(\tau)$$

$$X_2 = B(y)^{p'} s_{pq}(-\alpha_k \tau + E(y))$$

$$\Xi_1 = A_k(y)^{q'} [1 - s_{pq}(\tau)^q]^{1/p}$$

$$\Xi_2 = B(y)^{q'} [1 - s_{pq}(-\alpha_k \tau + E(y))^q]^{1/p}$$

where we put $\tau = \tau(t, y) = pA_k(y)^{m-p'-q'} t$.

Proposision 4.2 follows from a direct computation with Lemma 4.1 and with the differential equation (3.8).

§5. Uniformization of s_{pq}

We will treat the case $p+q \geq 7$ and construct a uniformization of the multi-valued function s_{pq} on $\mathbb{C}$. In the case $p+q \geq 7$, note that Proposition 3.5 implies

(5.1) $$p^{-1} + q^{-1} + m^{-1} < 1.$$

Notation 5.1 Let $\mathbb{D}$ be the the unit open disk in $\mathbb{C}$. We regard $\mathbb{D}$

as a hyperbolic plane with the *Poincaré metric*

$$ds^2=(1-|z|^2)^{-2}(dx^2+dy^2).$$

It is well-known that geodesics in $\mathbb{D}$ with respect to this metric consist of all (Euclidean) circles which are orthogonal to the circle at infinity $\{|z|=1\}$.

Definition 5.2. By (5.1) there exists a hyperbolic triangle with inner angles $p^{-1}\pi$, $q^{-1}\pi$ and $m^{-1}\pi$. We denote by $T_{pq}{}^{\#}$ the uniquely determined triangle with vertices 0, $\omega_{pq}{}^{\#}$, $\lambda_{pq}{}^{\#}$ so that

(5.2) $\quad \omega_{pq}{}^{\#}\in(0,1)$ and $\lambda_{pq}{}^{\#}\in\mathbb{D}\cap\{\mathrm{Im}(z)>0\}$

(5.3) $\quad \angle 0=q^{-1}\pi,\qquad \angle\omega_{pq}{}^{\#}=p^{-1}\pi,\qquad$ and $\qquad \angle\lambda_{pq}{}^{\#}=m^{-1}\pi.$

By Riemann's mapping theorem, there exists a map π which maps $T_{pq}{}^{\#}$ conformally onto the triangle T_{pq} defined by (3.3) [resp. maps $(T_{pq}{}^{\#})^{\sim}$ homeomorphically onto $T_{pq}{}^{-}$], where $^{\sim}$ denotes the closure in $\mathbb{D}$.

Notation 5.3. We denote by π_{pq} the conformal map $\pi:T_{pq}{}^{\#}\to T_{pq}$ with $(\pi(0),\pi(\omega_{pq}{}^{\#}),\pi(\lambda_{pq}{}^{\#}))=(0,\omega_{pq},\lambda_{pq})$. We denote the composition $s_{pq}\circ\pi_{pq}$ by σ_{pq}.

We consider the analytic [resp. meromorphic] continuation of π_{pq} [σ_{pq}] to $\mathbb{D}$. To do this we need to construct a polygon F_{pq} in $\mathbb{D}$, which is obtained by similar way of the construction (3.11) of F_{pq} in the case $p+q\leqq 6$.

Definition 5.4. Let $(T_{pq}{}^{\#})'$ be the reflection of $T_{pq}{}^{\#}$ with the geodesic $(-1,1)$, and we put $Q:=[T_{pq}{}^{\#}\cup(T_{pq}{}^{\#})']^{\sim}$. We denote by r_k [resp. ρ] the elliptic $2\pi/p$ [$2\pi/q$] rotation in $\mathbb{D}$ with center $\zeta_q{}^k\omega_{pq}{}^{\#}$ [the origin]. We define F_{pq} by

(5.4) $\quad F_{pq}:=$ the interior of $\left[\bigcup_{j=0}^{p-1}\bigcup_{k=0}^{q-1} r_k{}^j(\rho^k(Q))\right].$

Note that F_{pq} is a $2(p-1)q$-gon in $\mathbb{D}$. For each $n\in\mathbb{Z}$ satisfying $0\leqq n\leqq(p-1)q-1$, dividing n by $(p-1)$, we write

(5.5) $\quad n=(p-1)k+j\quad(0\leqq k,0\leqq j\leqq p-2).$

Definition 5.5. We define *sides* s_n and s_n' of F_{pq} and *side-pairing maps*

$$\{g_s\in\mathrm{Aut}(\mathbb{D});\ s\in\{s_n,s_n';0\leqq n\leqq(p-1)q-1\}\}$$

as follows:

(5.6) $\quad s_n = r_k^{j+1}(\rho^k([0,\lambda_{pq}^{\#}]))$, and

$\quad s_n' = r_{k+1}^{j+1}(\rho^k([0,\lambda_{pq}^{\#}]))$

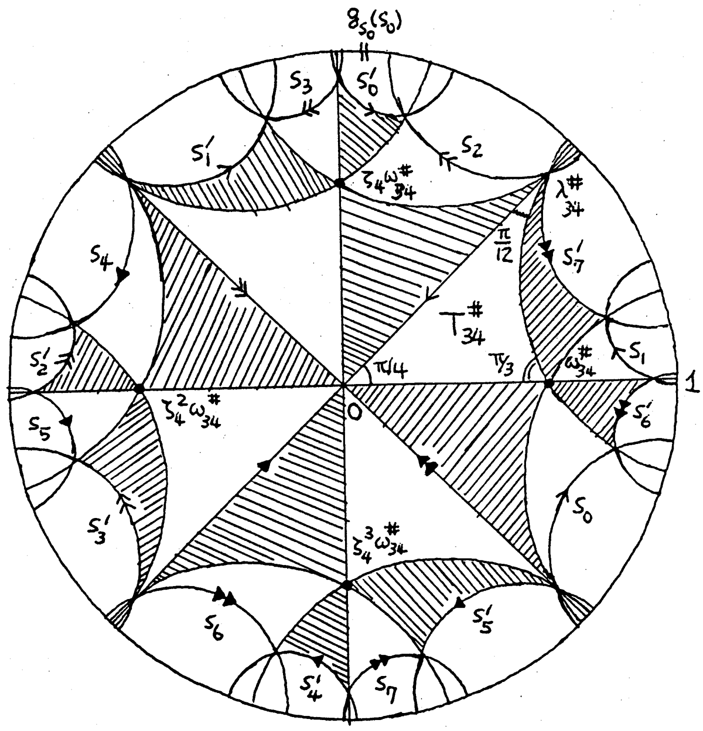
(5.7) $\quad g_s = r_{k+1}^{j+1} \circ r_k^{-j-1}$ if $s = s_n$, and

$\quad g_s = r_k^{j+1} \circ r_{k+1}^{-j-1}$ if $s = s_n'$,

where $[0,\lambda_{pq}^{\#}]$ denotes the geodesic segment joining 0 to $\lambda_{pq}^{\#}$.

We illustrate the polygon F_{pq} in the case $(p,q)=(3,4)$:

Figure 5.6. F_{34} with the side-pairing maps.



We recall the following general definition.

Definition 5.7. Let G be a subgroup of $\mathrm{Aut}(\mathbb{D})$. A subset Ω is called a *fundamental domain for* G if

(i) Ω is a domain.

(ii) The hyperbolic area of $\partial\Omega$ is 0.

(iii) $\mathbb{D} = \bigcup_{g \in G} g(\Omega^{\sim})$ (where $^{\sim}$ denotes the closure in $\mathbb{D}$).

(iv) $g \in G \setminus \{I\}$ implies $g(\Omega) \cap \Omega = \emptyset$.

Definition 5.8. A fundamental domain Ω for G is called *locally finite* if for any compact K in $\mathbb{D}$ only finitely many g in G satisfy $g(\Omega^{\sim}) \cap K \neq \emptyset$.

We now use the Poincaré's theorem originated by [3], which gives a sufficient condition for a polygon P with side-pairing maps

(5.8) $\{g_S;\ s \in \{\text{sides of } P\}\}$

to be a locally finite fundamental domain for the group generated by the side-pairing maps. To state the Poincaré's theorem we need a notion of a cycle of a vertex of P.

Definition 5.9. Let x be a vertex of P. We call a finite sequence $\{x_0, x_1, \cdots, x_{n-1}\}$ a *cycle* of x if

(i) $x_0 = x_n = x$, and $x_i \neq x_j$ for $0 \leqq i < j \leqq n-1$.

(ii) For any $1 \leqq j \leqq n$ there exist sides s_j and t_j of P so that

(5.9) $\{x_j\} = s_j \cap t_j$, $g_{S_j}(s_j) = t_{j-1}$ and $g_{S_j}(x_j) = x_{j-1}$.

We note that if $C(x) = \{x_0, x_1, \cdots, x_{n-1}\}$ is a cycle of x then every x_j is a vertex of P, and that $C^{-1}(x) = \{x_n, x_{n-1}, \cdots, x_1\}$ is also a cycle of x. Further, there is no cycle of x different from $C(x)$ and $C^{-1}(x)$.

Theorem 5.10 (Poincaré's theorem of a restricted type). *Let* P *be a relatively compact polygon in* $\mathbb{D}$ *with side-pairing maps* (5.8). *Assume the following condition* (*angle condition*) (5.10) *for any vertex* x*: let* $C(x) = \{x_0, x_1, \cdots, x_{n-1}\}$ *be a cycle of* x, *and let* θ_j *be the inner angle of* P *at the vertex* x_j. *In this situation, there exists a* $N = N(C(x)) \in \mathbb{N}$ *so that*

$$\sum_{j=0}^{n-1} \theta_j = 2\pi/N. \tag{5.10}$$

Then P *is a locally finite fundamental domain for the group which is generated by the collection* (5.8).

For the proof of Theorem 5.10 see [1] or [3].

We want now to apply Theorem 5.10 to our F_{pq} with the side-pairing maps (5.7). To do this we must verify the following lemma.

Lemma 5.11. *The polygon* F_{pq} *with the side-pairing maps* (5.7) *satisfies the angle condition* (5.10) *with* $N(C(x)) = 1$ *for all vertices* x *of* F_{pq}.

Proof. Let x be a vertex of F_{pq}. Then there exists a uniquely determined $n \in \{0, 1, \cdots, (p-1)q-1\}$ such that x lies in s_n. There are two cases which can occur:

Case 1. $\quad x = r_k^{j+1}(\rho^k(\lambda_{pq}^{\#}))$,

Case 2. $\quad x = r_k^{j+1}(\rho^k(0)) = r_k^{j+1}(0)$,

where j and k are related with n by (5.5).

We will first show the following claim:

Claim 5.12. *Let* n' *be the uniquely determined integer by*

(5.11) $$x' := g_{s_n}(x) \in s_{n'}.$$

Then it follows that

(5.12) $\quad n' \equiv n + p \mod (p-1)q$ *in the case* 1,

(5.13) $\quad n' \equiv n + p - 1 \mod (p-1)q$ *in the case* 2.

Proof of Claim 5.12. We first show (5.12). By the definition (5.7) of the side-pairing maps, we have

$$x' = r_{k+1}^{j+1} r_k^{-j-1}(r_k^{j+1}(\rho^k(\lambda_{pq}^{\#}))) = r_{k+1}^{j+1}(\rho^k(\lambda_{pq}^{\#})).$$

Since $\rho^k(\lambda_{pq}^{\#}) = r_{k+1}(\rho^{k+1}(\lambda_{pq}^{\#}))$, we get

$$x' = r_{k+1}^{j+2}(\rho^{k+1}(\lambda_{pq}^{\#})).$$

Thus, by (5.6), $x' \in s_{n'}$ is equivalent to

$$n' \equiv (p-1)(k+1) + j + 1 \mod (p-1)q,$$

which implies

$$n' - n \equiv (p-1)(k+1) + j + 1 - [(p-1)k + j] = p.$$

Next we show (5.13). By (5.7) we also have

$$x' = r_{k+1}^{j+1} r_k^{-j-1}(r_k^{j+1}(0)) = r_{k+1}^{j+1}(\rho^{k+1}(0)).$$

So we get by (5.6) that $n' \equiv (p-1)(k+1) + j \mod (p-1)q$, which implies

$$n' - n \equiv (p-1)(k+1) + j - [(p-1)k + j] = p - 1.$$

This completes the proof of Claim 5.12. □

We continue the proof of Lemma 5.11. Let $C(x) = \{x_0, \cdots, x_{\nu-1}\}$ be the cycle of $x = x_0$. In the case 1, Claim 5.12 shows that $\nu = \# C(x)$ is the minimum of $\mu \in \mathbb{N}$ satisfying

(5.14) $$n + \mu p \equiv n \mod (p-1)q.$$

Since (5.14) is equivalent to $\mu p' \in (p-1)q'\mathbb{Z}$, the coprimeness of p' and $(p-1)q'$ implies $\mu \in (p-1)q'\mathbb{Z}$. Thus we get

$$\nu = \min[(p-1)q'\mathbb{N}] = (p-1)q'$$

Further, a vertex x_h $(1 \leqq h \leqq \nu = (p-1)q')$ in $C(x)$ coincides with $\rho^k(\lambda_{pq}{}^{\#})$ for some $k \in \{0, 1, \cdots, q-1\}$ if and only if h satisfies $n + hp \in (p-1)\mathbb{Z}$, which is equivalent to

(5.15) $$j + h \in (p-1)\mathbb{Z}$$

where $n = (p-1)k + j$. Since (5.15) has q'-solutions h, we have

$$\#\{x_h;\ \theta_h = 4m^{-1}\pi\} = q' \text{ and}$$

$$\#\{x_h;\ \theta_h = 2m^{-1}\pi\} = \nu - q' = (p-2)q'.$$

Thus we deduce in the case 1 that

$$\sum_{j=0}^{\nu-1} \theta_j = (4m^{-1}\pi)q' + (2m^{-1}\pi)(p-2)q'$$
$$= 2\pi m^{-1}pq' = 2\pi.$$

In the case 2, Claim 5.12 shows that $\nu = \#C(x)$ is the minimum of $\mu \in \mathbb{N}$ satisfying

(5.16) $$n + \mu(p-1) \equiv n \mod (p-1)q.$$

Then it is trivial that $\nu = q$, so we deduce in the case 2 that

$$\sum_{j=0}^{\nu-1} \theta_j = (2q^{-1}\pi)q = 2\pi.$$

It completes the proof of Lemma 5.11. □

By virtue of Lemma 5.11 we can apply Theorem 5.10 (Poincaré's theorem) to the polygon F_{pq} with the side-pairing maps (5.7), and have the following proposition.

Proposition 5.13. *The polygon* F_{pq} *defined by* (5.4) *is a locally finite fundamental domain for the group generated by the side-pairing maps* (5.7).

Notation 5.14. We denote by G_{pq} the group generated by the side-pairing maps (5.7).

As a consequence of Proposition 5.13 we have the following uniformization of s_{pq}.

Corollary 5.15. *The following* (i)~(iv) *hold:*

(i) σ_{pq} *is meromorphic on* $\mathbb{D}$, *and is* G_{pq}*-invariant.*

(ii) π_{pq} *is holomorphic on* $\mathbb{D}$, *and for any* g *in* G_{pq} *the function* $c(g)(z) := \pi_{pq}(gz) - \pi_{pq}(z)$ *is constant on* $\mathbb{D}$.

(iii) *The map* $g \to c(g)$ *is a group homomorphism from* G_{pq} *into the additive group* $(\mathbb{C}, +)$.

(iv) $\sigma_{pq}(z) = s_{pq}(\pi_{pq}(z))$ *on* $\mathbb{D}$, *that is, the following diagram is commutative:*

(5.17)
$$\begin{tikzcd} \mathbb{D} \arrow[r, "\sigma_{pq}"] \arrow[d, "\pi_{pq}"'] & \hat{\mathbb{C}} \\ \mathbb{C} \arrow[ur, dashed, "s_{pq}"'] & \end{tikzcd}$$

§6. A Picard Type Theorem

In this section we give a Picard type theorem for a uniformization of the equation (6.2) below.

Let $\Phi_k = (X; \Xi)$ be the Hamilton flow, and we consider the following equation in $(t, y) \in \mathbb{C} \times \Omega_1$:

(6.1) $$X_j(t, y) = x_j \quad (j = 1, 2)$$

where Ω_1 is the domain chosen as in Lemma 4.1. By Proposition 4.2, putting $\tau = pA_k(y)^{m-p'-q'} t$, (6.1) can be written as

(6.2) $$\begin{aligned} \tilde{X_1}(\tau, y) &= A_k(y)^{p'} s_{pq}(\tau) &= x_1 \\ \tilde{X_2}(\tau, y) &= B(y)^{p'} s_{pq}(-\alpha_k \tau + E(y)) &= x_2 \end{aligned}$$

Since in the case $p + q \geqq 7$ the function s_{pq} is multi-valued, we introduce the following uniformization of the map $\tilde{X}$.

Notation 6.1. Let us put

(6.3) $$\Sigma_{pq} := \begin{cases} \mathbb{D} & \text{if } p + q \geqq 7 \\ \mathbb{C} & \text{if } 5 \leqq p + q \leqq 6. \end{cases}$$

In the case $\Sigma_{pq} = \mathbb{D}$, the maps π_{pq} and σ_{pq} are already defined in Notation 5.3. In the case $\Sigma_{pq} = \mathbb{C}$, we denote the identity map on $\mathbb{C}$ by π_{pq}, and we put $\sigma_{pq} = s_{pq}$.

Using this notation, we have $\pi_{pq} \in \mathcal{O}(\Sigma_{pq})$ and $\sigma_{pq} \in \mathrm{Mero}(\Sigma_{pq})$ so that

(6.4) $$\sigma_{pq}(z) = s_{pq}(\pi_{pq}(z)) \quad \text{on } \Sigma_{pq}.$$

Definition 6.2. For the constant α_k and $E \in \mathcal{O}(\Omega_1)$ in (6.2), we define a surface $M_{pq}(\alpha_k)$ by

(6.5) $$M_{pq}(\alpha_k) := \{(z_1, z_2; y) \in \Sigma_{pq}{}^2 \times \Omega_1 : \\ -\alpha_k \pi_{pq}(z_1) + E(y) = \pi_{pq}(z_2)\}.$$

We also define a map $X^{\#} : M_{pq}(\alpha_k) \to \mathbb{C}^2$ by

(6.6) $$X^{\#}(z_1, z_2; y) = (A_k(y)^{p'} \sigma_{pq}(z_1),\ B(y)^{p'} \sigma_{pq}(z_2)).$$

Finally we define a map $P : M_{pq}(\alpha_k) \to \mathbb{C} \times \Omega_1$ by

(6.7) $$P(z_1, z_2; y) = (\pi_{pq}(z_1),\ y).$$

Remark 6.3. The identity (6.4) implies the following identity:

(6.8) $$X^{\#}(z_1, z_2; y) = X^{\sim}(P(z_1, z_2; y)).$$

Proof. Indeed, it follows from (6.4) and (6.5) that

$$X_1{}^{\sim}(P(z_1, z_2; y)) = A_k(y)^{p'} s_{pq}(\pi_{pq}(z_1))$$
$$= A_k(y)^{p'} \sigma_{pq}(z_1) = X_1{}^{\#}(z_1, z_2; y),$$
$$X_2{}^{\sim}(P(z_1, z_2; y)) = B(y)^{p'} s_{pq}(-\alpha_k \pi_{pq}(z_1) + E(y))$$
$$= B(y)^{p'} s_{pq}(\pi_{pq}(z_2)) = B(y)^{p'} \sigma_{pq}(z_2)$$
$$= X_2{}^{\#}(z_1, z_2; y)$$ □

By virtue of Remark 6.3, the equation (6.2) also has the following uniformization

(6.9) $$X_1{}^{\#}(z_1, z_2; y) = A_k(y)^{p'} \sigma_{pq}(z_1) = x_1$$
$$X_2{}^{\#}(z_1, z_2; y) = B(y)^{p'} \sigma_{pq}(z_2) = x_2.$$

Notation 6.4. We put $(G_{pq})_* = \{c(g) \in \mathbb{C}\,;\, g \in G_{pq}\}$, where $c(g)$ is the constant defined by $c(g) = \pi_{pq}(gz) - \pi_{pq}(z)$, $z \in \Sigma_{pq}$.

By Corollary 5.15 (iii) $(G_{pq})_*$ forms an additive subgroup of $\mathbb{C}$.

The following lemma is fundamental to solve (6.9).

Lemma 6.5. *The vector sum*

$$\alpha_k (G_{pq})_* + (G_{pq})_* = \{\alpha_k x + y\,;\, x, y \in (G_{pq})_*\}$$

is dense in $\mathbb{C}$.

We omit the proof of Lemma 6.5, which is obtained by the fact that α_k lies in $\mathbb{C} \setminus \mathbb{Q}(\zeta_m)$ (see Lemma 4.1 and [A.3]), and that $c(g)/\omega_{pq}$ lies in $\mathbb{Z}[\zeta_m]$.

Using Lemma 6.5, we can show the following Picard type theorem for the map $X^{\#}: M_{pq}(\alpha_k) \to \mathbb{C}^2$.

Proposition 6.6. *There exists a relatively compact subdomain* Ω_2 *of* Ω_1 *such that the following* (i) *and* (ii) *hold:*
(i) *There exists an open neighborhood* V *of the origin in* $\mathbb{C}^2$ *such that for any* $x \in V$ *we can find a distinct sequence* $\{(z_{1\nu}, z_{2\nu}; y_\nu); \nu \in \mathbb{N}\}$ *in* $M_{pq}(\alpha_k) \cap [\Sigma_{pq}{}^2 \times \Omega_2]$ *satisfying*

(6.10) $$X^{\#}(z_{1\nu}, z_{2\nu}; y_\nu) = x \quad \textit{for any } \nu \in \mathbb{N}.$$

(ii) *Moreover, if* $x \in V \setminus \{0\}$ *then* $\{(z_{1\nu}, z_{2\nu}; y_\nu)\}$ *has the following property: for any* $\mu \neq \nu$,

(6.11) $$z_{1\mu} \in G_{pq}(z_{1\nu}) \quad \textit{and} \quad z_{2\mu} \in G_{pq}(z_{2\nu})$$

are not compatible, where $G_{pq}(z)$ *denotes the* G_{pq}*-orbit containing* $z \in \Sigma_{pq}$.

Proof of (i). Let us reall the facts shown in § 4:

(6.12) $$A_k(y)^m = -a\,\psi(y) \neq 0, \quad B(y)^m = \psi(y) \neq 0 \quad \text{on } \Omega_1,$$

(6.13) $$E(y) = (\tau_{pq} | V_q)(y / B(y)^{p'}),$$

where V_q is the simply connected domain given by (4.2). Since $\{y/B(y)^{p'}\}^q = y^q/\psi(y)$ is not constant by [A.2], we get

(6.14) $$E(y) \text{ is not constant on } \Omega_1.$$

We also remark that

(6.15) $$\psi(y) \text{ is not constant on } \Omega_1.$$

Indeed, if we assume $\psi(y) \equiv c$ then $c \neq 0$, because $\psi(y) \equiv 0$ implies $\phi'(y)^p / y^q \equiv -1$ which violates [A.2]. On the other hand, $\psi(y) \equiv c \neq 0$ implies $\phi'(y) \equiv [c - y^q]^{1/p}$ which contradicts $\phi'(y) \in \mathcal{O}(\mathbb{C})$. Thus we get (6.15).

By (6.14) and (6.15), there exists a relatively compact subdomain Ω_2 of Ω_1 so that

(6.16) $$\min_{y \in \overline{\Omega_2}} |E'(y)| > 0 \quad \text{and} \quad \min_{y \in \overline{\Omega_2}} |\psi'(y)| > 0.$$

Since $\psi(y) \neq 0$ on Ω_1, we also have

$$\min_{y \in \overline{\Omega_2}} |\psi(y)| > 0.$$

Thus (6.12) yields

(6.17) $$\min_{y\in\overline{\Omega_2}} |A_k(y)| > 0 \quad\text{and}\quad \min_{y\in\overline{\Omega_2}} |B(y)| > 0.$$

Then, by (6.17) and the former of (6.16), we get

(6.18) $$\min\{\min_{y\in\overline{\Omega_2}} |A_k(y)|,\ \min_{y\in\overline{\Omega_2}} |B(y)|,\ \min_{y\in\overline{\Omega_2}} |E'(y)|\} > 0.$$

We denote the left hand side of (6.18) by ε. We consider the following function H:

(6.19) $$H(y;x_1,x_2) := -\alpha_k(\tau_{pq}|V_q)(x_1/A_k(y)^{p'}) + E(y) + (\tau_{pq}|V_q)(x_2/B(y)^{p'})$$
$$\text{for}\quad y\in\Omega_2 \quad\text{and}\quad |x_j| < \varepsilon^{p'} \quad (j=1,2).$$

Since $y\in\Omega_2$ and $|x_j|<\varepsilon^{p'}$ yield $|x_1/A_k(y)^{p'}|$, $|x_2/B(y)^{p'}| < 1$, we have $x_1/A_k(y)^{p'}$, $x_2/B(y)^{p'} \in V_q$, so $H(y;x_1,x_2)$ is well-defined. Moreover, since

$$(\partial/\partial y)H = -\alpha_k(\tau_{pq}|V_q)'(x_1/A_k(y)^{p'})x_1(d/dy)[A_k^{-p'}] + E'(y) + (\tau_{pq}|V_q)'(x_2/B(y)^{p'})x_2(d/dy)[B^{-p'}],$$

we can find a small $\delta\in(0,\varepsilon^{p'})$ such that $|x_j|<\delta$ yield

(6.20) $$|(\partial/\partial y)H(y;x_1,x_2)|$$
$$\geqq \min_{y\in\overline{\Omega_2}} |E'(y)| - \delta(\max_{|z|\leqq\delta/\varepsilon^{p'}} |(\tau_{pq}|V_q)'(z)|)$$
$$\times \max_{y\in\overline{\Omega_2}} (|\alpha_k||(d/dy)[A_k^{-p'}]| + |(d/dy)[B^{-p'}]|)$$
$$\geqq \varepsilon/2.$$

We put $V := \{x\in\mathbb{C}^2 ; |x_j|<\delta\}$.

For any $x\in V$ we constract a sequence $\{(z_{1\nu},z_{2\nu};y_\nu);\nu\in\mathbb{N}\}$ of solutions of (6.10) as follows. Since (6.20) implies that, for any fixed $x\in V$, the function $y \to H(y;x)$ is not constant on Ω_2, the image $W(x)=\{H(y;x);y\in\Omega_2\}$ is a non-empty open set in $\mathbb{C}$. Then, by Lemma 6.5, $W(x)\cap[\alpha_k(G_{pq})_*+(G_{pq})_*]$ contains infinitely many elements, so we can choose sequences $\{y_\nu\}$ in Ω_2, and $\{g_\nu\}$, $\{h_\nu\}$ in G_{pq} such that

(6.21) $$H(y_\nu;x) = -\alpha_k c(g_\nu) + c(h_\nu) \quad\text{with the property}$$
$$H(y_\nu;x) \neq H(y_\mu;x) \quad\text{for any}\quad \nu\neq\mu.$$

Taking subsequences if necessary, we may assume that there is a $y_0 \in\overline{\Omega_2}$ so that $y_\nu \to y_0$ ($\nu\to\infty$). Now we define a sequence

$\{(z_{1\nu}, z_{2\nu}); \nu \in \mathbb{N}\}$ in $\Sigma_{pq}{}^2$ by

(6.22) $\quad z_{1\nu} := g_\nu((\sigma_{pq}|F_{pq}{}^*)^{-1}(x_1/A_k(y_\nu)^{p'}))$

$\quad z_{2\nu} := h_\nu((\sigma_{pq}|F_{pq}{}^*)^{-1}(x_2/B(y_\nu)^{p'}))$,

where $F_{pq}{}^*$ is, using Q and ρ in Definition 5.4, defined by

(6.23) $\quad F_{pq}{}^* :=$ the interior of $[\bigcup_{k=0}^{q-1} \rho^k(Q)]$.

We note that $F_{pq}{}^* \subset F_{pq}$ and the following diagram commutes:

(6.24)
$$\begin{tikzcd} F_{pq}{}^* \arrow[r, "\sigma_{pq}|F_{pq}{}^*", "\sim"'] \arrow[d, "\pi_{pq}|F_{pq}{}^*"', "\wr"] & V_q \arrow[dl, "\tau_{pq}|V_q", "\sim"'] \\ \tau_{pq}(V_q) & \end{tikzcd}$$

Then the G_{pq}-invariance of σ_{pq} yields

$$X_1{}^\#(z_{1\nu}, z_{2\nu}; y_\nu) = A_k(y_\nu)^{p'} \sigma_{pq}(z_{1\nu})$$
$$= A_k(y_\nu)^{p'} \sigma_{pq}((\sigma_{pq}|F_{pq}{}^*)^{-1}(x_1/A_k(y_\nu)^{p'})) = x_1,$$
$$X_2{}^\#(z_{1\nu}, z_{2\nu}; y_\nu) = B(y_\nu)^{p'} \sigma_{pq}(z_{2\nu})$$
$$= B(y_\nu)^{p'} \sigma_{pq}((\sigma_{pq}|F_{pq}{}^*)^{-1}(x_2/B(y_\nu)^{p'})) = x_2.$$

Moreover, (6.24) and the property $\pi_{pq}(gz) = \pi_{pq}(z) + c(g)$ imply

$$-\alpha_k \pi_{pq}(z_{1\nu}) + E(y_\nu) - \pi_{pq}(z_{2\nu})$$
$$= -\alpha_k[\pi_{pq}((\sigma_{pq}|F_{pq}{}^*)^{-1}(x_1/A_k(y_\nu)^{p'})) + c(g_\nu)] + E(y_\nu)$$
$$-[\pi_{pq}((\sigma_{pq}|F_{pq}{}^*)^{-1}(x_2/B(y_\nu)^{p'})) + c(h_\nu)]$$
$$= H(y_\nu; x) - [\alpha_k c(g_\nu) + c(h_\nu)]$$
$$= 0.$$

Thus, the assertion (i) is proved.

Proof of (ii). It suffices to show that if there exist ν and μ with $\nu \neq \mu$ so that (6.11) are compatible, then $x = 0$. Since $z_{1\mu} \in G_{pq}(z_{1\nu})$, there exists a $g \in G_{pq}$ so that $z_{1\mu} = g(z_{1\nu})$, so (6.22) yields

$$g_\mu((\sigma_{pq}|F_{pq}{}^*)^{-1}(x_1/A_k(y_\mu)^{p'}))$$
$$= g g_\nu((\sigma_{pq}|F_{pq}{}^*)^{-1}(x_1/A_k(y_\nu)^{p'})).$$

Since the image of $(\sigma_{pq}|F_{pq}{}^*)^{-1}$ lies in F_{pq} which is a fundamental domain for G_{pq} by Proposition 5.13, we get $g_\mu = g g_\nu$ and

$(\sigma_{pq}|F_{pq}^*)^{-1}(x_1/A_k(y_\mu)^{p'}) = (\sigma_{pq}|F_{pq}^*)^{-1}(x_1/A_k(y_\nu)^{p'})$.

Then the injectivity of $(\sigma_{pq}|F_{pq}^*)^{-1}$ implies

(6.25) $$x_1/A_k(y_\mu)^{p'} = x_1/A_k(y_\nu)^{p'}.$$

If we assume $x_1 \neq 0$ then $A_k(y_\mu)^{p'} = A_k(y_\nu)^{p'}$. Since we get

$$(d/dy)[A_k(y)^{p'}] = p' A_k(y)^{p'-1} A_k'(y) \neq 0$$

on Ω_2^-, which is a consequence of (6.12),(6.17) and the latter of (6.16), there exists an open neighborhood U of $y_0 = \lim y_\nu$ so that $y \to A_k(y)^{p'}$ is injective on U. Thus, taking a subsequence if necessary, we may assume that $y_\nu \in U$ for all ν. Then we have $A_k(y_\mu)^{p'} \neq A_k(y_\nu)^{p'}$, so $x_1 \neq 0$ is impossible. Hence we get $x_1 = 0$. Since the similar argument also yields $x_2 = 0$, we get the assertion (ii). It completes the proof. □

§7. Proof of Theorem 1.3

Now we give a proof of our main result (Theorem 1.3) in this last section. The Picard type theorem (Proposition 6.6) shows that the fiber $(X^\#)^{-1}(x)$ is an infinite set in $M_{pq}(\alpha_k)$ for $x \in V$.

On the other hand, we can show the following fact.

Theorem 7.1. *Let* Ω_2 *be the relatively compact subdomain chosen as in Proposition 6.6. We put* $M_{pq}^*(\alpha_k) := M_{pq}(\alpha_k) \cap (\Sigma_{pq}^2 \times \Omega_2)$. *Then, for any* $k = 1, 2, \cdots, p$, *the surface* $M_{pq}^*(\alpha_k)$ *is a non-singular connected surface.*

Partial proof. The non-singularity follows from the former of (6.16). The connectivity of $M_{pq}^*(\alpha_k)$ in the case $p+q \leqq 6$ is easily verified, because $\pi_{pq} = \mathrm{id}$ on $\Sigma_{pq} = \mathbb{C}$ implies that the surface $M_{pq}^*(\alpha_k) = \{(z_1, z_2; y) \in \mathbb{C}^2 \times \Omega_2 ; -\alpha_k z_1 + E(y) = z_2\}$ is a continuous image of the connected set $\mathbb{C} \times \Omega_2$. But in the case $p+q \geqq 7$ it needs a long proof with full use of $\alpha_k \in \mathbb{C} \setminus \mathbb{Q}(\zeta_m)$ to show that $M_{pq}^*(\alpha_k)$ is connected. So, we omit it here. □

To show Theorem 1.3, we recall the following diagram, which is introduced in Definition 6.2 and Remark 6.3.

$$(7.1)\qquad \begin{tikzcd}
(z;y)\in M_{pq}{}^*(\alpha_k) \arrow[r, "X^\#"] \arrow[d, "P"] & \mathbb{C}^2 \\
(\pi_{pq}(z_1), y) \in \mathbb{C}_\tau \times \Omega_2 \arrow[ur, dashed, "X^\sim"] \arrow[d] & \\
\left(\frac{\pi_{pq}(z_1)}{pA_k(y)^{m-p'-q'}}, y\right) \in \mathbb{C}_t \times \Omega_2 \arrow[uur, dashed, "X"'] &
\end{tikzcd}$$

Let $(X^\#)^{-1}$ be the multi-valued analytic inverse of $X^\#$, and let V be the open neighborhood of the origin in $\mathbb{C}^2$ chosen as in Proposition 6.6. For any fixed $x\in V$ and for two points $(z_\nu;y_\nu)$ and $(z_\mu;y_\mu)$ in the fiber $(X^\#)^{-1}(x)$, by Theorem 7.1, there exists a path $\Gamma_{\nu\mu}$ in $M_{pq}{}^*(\alpha_k)$ which joins $(z_\nu;y_\nu)$ to $(z_\mu;y_\mu)$. Since the 2-form $\beta:=dX_1{}^\#\wedge dX_2{}^\#$ does not vanish identically on $M_{pq}{}^*(\alpha_k)$, the complement of $\beta^{-1}(0)$ is open dense and connected in $M_{pq}{}^*(\alpha_k)$. Thus we can choose $\Gamma_{\nu\mu}\setminus\{(z_\nu;y_\nu),(z_\mu;y_\mu)\}$ in the complement of $\beta^{-1}(0)$. So, the two germs ρ_ν and ρ_μ of $(X^\#)^{-1}$ at x with $\rho_\nu(x)=(z_\nu;y_\nu)$, $\rho_\mu(x)=(z_\mu;y_\mu)$ can be continued analytically each other along the path $X^\#(\Gamma_{\nu\mu})$ in $\mathbb{C}^2$. We put

$$(7.2)\qquad \begin{aligned}(X^\#)^{-1}(x)&=(z(x);y(x))\\ X^{-1}(x)&=(t(x),y(x))\end{aligned}$$

where $t(x)$ is given by

$$(7.3)\qquad t(x)=\pi_{pq}(z_1(x))/[pA_k(y(x))^{m-p'-q'}].$$

Note that, if we restrict analytic continuations of $(X^\#)^{-1}$ to the continuations along paths of the type $X^\#(\Gamma_{\nu\mu})$, then $y(x)\in\Omega_2$, so $t(x)$ is well-defind by (7.3).

Let $Z_k(t,y)$ be the solution of

$$(7.4)\qquad \begin{aligned}(\partial/\partial t)Z&=\Xi_1[(\partial/\partial\xi_1)F](\Phi_k)+\Xi_2[(\partial/\partial\xi_2)F](\Phi_k)\\ &=p\Xi_1{}^p+ap\Xi_2{}^p\\ Z(0,y)&=\phi(y)\end{aligned}$$

where $\Phi_k(t,y)=(X(t,y);\Xi(t,y))$ is the Hamilton flow issuing from $\rho_k(y)$ at $t=0$. We put

$$(7.5)\qquad w_k(x):=Z_k(X^{-1}(x)).$$

By the theory of characteristic strips in §2, $w_k(x)$ is an analytic continuation of the solution $u_k(x;\Omega)$ of the Cauchy problem (1.1), so all germs of w_k are contained in germs of the maximal continuation $u_k^*(x;\Omega)$.

Now we assume that the conclusion of Theorem 1.3 is false, that is, $u_k^*(x;\Omega)$ is finitely many-valued. Then w_k is also finitely many-valued, so are $(\partial/\partial x_1)w_k$ and $(\partial/\partial x_2)w_k$. By (2.5) in Proposition 2.1 and by the uniqueness of continuations, we have

(7.6) $$(\partial/\partial x_j)w_k(x)=\Xi_j(X^{-1}(x)) \quad \text{for} \quad j=1,2.$$

Since

$$\Xi_1(t,y)^p+X_1(t,y)^q=\Xi_1(0,y)^p+X_1(0,y)^q=-a\,\psi(y),$$

we have $[(\partial/\partial x_1)w_k(x)]^p+x_1^q=-a\,\psi(y(x))$, which implies that $\psi(y(x))$ is finitely many-valued. Then the relations

$$A_k(y)^m=-a\,\psi(y) \quad \text{and} \quad B(y)^m=\psi(y)$$

yield that both $A_k(y(x))$ and $B(y(x))$ are finitely many-valued functions. Then, by the equations

$$x_1=X_1^{\#}(z(x);y(x))=A_k(y(x))^{p'}\,\sigma_{pq}(z_1(x))$$

$$x_2=X_2^{\#}(z(x);y(x))=B(y(x))^{p'}\,\sigma_{pq}(z_2(x)),$$

we deduce

(7.7) $\sigma_{pq}(z_1(x))$ and $\sigma_{pq}(z_2(x))$ are finitely many-valued.

From now on we fix $x\in V\setminus\{0\}$, and let $\{(z_{1\nu},z_{2\nu};y_\nu)\}$ be the sequence in Proposition 6.6. Then (7.7) yields that the set

(7.8) $\{\sigma_{pq}(z_{j\nu});\nu\in\mathbb{N}\}$ is finite for $j=1,2$.

By (7.8), taking a subsequences of $\{z_{1\nu}\}$ and $\{z_{2\nu}\}$ if necessary, we may assume that there exist constants c_j $(j=1,2)$ so that

(7.9) $$\sigma_{pq}(z_{j\nu})=c_j \quad (j=1,2).$$

Since the restriction $\sigma_{pq}|F_{pq}:F_{pq}\to\hat{\mathbb{C}}$ is a p-to-1 map, and since σ_{pq} is G_{pq}-invariant, we deduce from (7.9) the following inequality:

(7.10) $$\#[\{z_{1\nu}\}/G_{pq}]\leqq p.$$

Then, taking a subsequences of $\{z_{1\nu}\}$ if necessary, we may assume that $\{z_{1\nu}\}$ is contained in the same G_{pq}-orbit $G_{pq}(z_{11})$.

Finally we consider the finite sequence $\{z_{2\nu};1\leqq\nu\leqq p+1\}$. Since (7.9) implies the inequality

$$\#[\{ z_{2\nu} ; 1 \leqq \nu \leqq p+1 \}/G_{pq}] \leqq p$$

as similar as (7.10), there exist $\nu, \mu \in \{1, 2, \cdots, p+1\}$ with $\nu \neq \mu$ such that $z_{2\mu} \in G_{pq}(z_{2\nu})$. Thus we conclude that there exist ν and μ with $\nu \neq \mu$ such that

$$z_{1\mu} \in G_{pq}(z_{1\nu}) \text{ and } z_{2\mu} \in G_{pq}(z_{2\nu})$$

are compatible. This contradicts the assertion (ii) of Proposition 6.6. Thus the maximal analytic continuation $u_k^*(x;\Omega)$ is an infinitely many-valued function. It completes the proof of Theorem 1.3.

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